Summary: The Feynman-Kac formula
exercises

The Feynman-Kác formula is very helpful to solve partial differential equation problems, as the following:

Find a function $F: \mathbb{R}^{+} \times \mathbb{R} \to \mathbb{R}$ such that it is solution of the following partial differential equation:

$$\forall t \in [0,T] \qquad \frac{\partial F(t,x)}{\partial t} + \underline{\mu(\ \ x)} \frac{\partial F(t,x)}{\partial x} + \frac{1}{2}\underline{\sigma^2(\ \ x)} \frac{\partial^2 F(t,x)}{\partial x^2} = 0 \qquad (*)$$

boundary condition: $F(T,x) = \underline{g(x)}$

This is known as the Cauchy problem.

$\boxed{\text{This problem is completely deterministic}}$

But we are going to use stochastic processes and random variables to solve it!

<u>Derivation of the result</u>

1°: consider an "artificial" process, $\{x(t), t \geq 0\}$ such that this process is solution of the following stochastic differential equation

$$dx(t) = \mu(x(t))\,dt + \sigma(x(t))\,dw(t)$$

2°: we apply Itô's formula to $F(t, x(t))$:

$$dF(t, x(t)) = \left[\frac{\partial F}{\partial t} + \mu(x(t))\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(x(t))\frac{\partial^2 F}{\partial x^2}\right]dt$$

$$+ \sigma(x(t))\,dw(t)$$

Although we don't know how much is F we know that F is solution of (*)

of the following partial differential equation:

$$\forall t\in[0,T] \qquad \frac{\partial F(t,x)}{\partial t} + \mu(x)\frac{\partial F(t,x)}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 F(t,x)}{\partial x^2} = 0 \qquad (*)$$

Hence F is such that:

$$d\,F(t, X(t)) = \sigma(X(t))\,dW(t)$$

3^{rd}: We compute $\int_t^T dF(s,X(s))$

$$\int_t^T dF(s,X(s)) = \int_t^T \sigma(X(s))\,dW(s)$$

$$\Leftrightarrow$$

$$F(T, X(T)) - F(t, X(t)) = \int_t^T \sigma(X(s))\,dW(s)$$

4^{th} use the boundary condition:

$$F(T,x) = g(x) \qquad\qquad g(X(T)) - F(t, X(t)) = \int_t^T \sigma(X(s))\,dW(s)$$

5^{th} Next we compute $E[\quad | X(t)=x]$

$$E[g(X(T)) \mid X(t)=x] - \overbrace{E[F(t,X(t)) \mid X(t)=x]}^{F(t,x)}$$

$$= E\left[\int_t^T \cancel{\sigma(X(s))\,dW(s)} \mid X(t)=x\right] \quad (i)$$

$$\shortparallel \; 0$$

6^{th}: FK formula:

$$\boxed{F(t,x) = E[g(X(T)) \mid X(t)=x]}$$

(i) $E\left[\int_t^T \sigma(X(s))\,dW(s) \mid \underbrace{X(t)=x}_{\mathcal{F}_t^W}\right] =$

$$\overset{\text{independence}}{=} E\left[\int_t^T \sigma(X(s))\,dW(s)\right] = 0$$

Example

use Feynman-Kac

1. Use a stochastic representation result in order to solve the following boundary value problem in the domain $[0, T] \times \mathbb{R}$:

$$\frac{\delta F}{\delta t} + \overset{\mu}{\mu x} \frac{\delta F}{\delta x} + \overset{\frac{1}{2}\sigma^2}{\frac{1}{2}\sigma^2 x^2} \frac{\delta^2 F}{\delta x^2} = 0$$

$F(t, x) = E[\ln X^2(T) \mid X(t) = x]$

with $F(T, x) = \ln(x^2)$, where μ and σ are known.

$dX(\tau) = \mu(X(\tau))\,d\tau + \sigma(X(\tau))\,dW(\tau)$

$\mu(X(\tau)) = \mu X(\tau)$

$\sigma(X(\tau)) = \sigma X(\tau)$

$\Rightarrow \quad dX(\tau) = \mu X(\tau)\,d\tau + \sigma X(\tau)\,dW(\tau)$

solution: $X(T) = X(t)\, e^{(\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma(W(T) - W(t))}$

FK: $F(t, x) = E[\ln(X^2(T)) \mid X(t) = x]$

$X^2(T) = X^2(t)\, e^{2(\mu - \frac{1}{2}\sigma^2)(T-t) + 2\sigma(W(T) - W(t))}$

$\ln X^2(T) = 2(\mu - \frac{1}{2}\sigma^2)(T-t) + 2\sigma(W(T) - W(t)) + 2\ln X(t)$

Thus: $F(t, x) = E[2(\mu - \frac{1}{2}\sigma^2)(T-t) + 2\sigma(W(T) - W(t)) +$

$+ 2\ln X(t) \mid X(t) = x]$

$W(t)$

$= 2(\mu - \frac{1}{2}\sigma^2)(T-t) + 2\sigma E[W(T) - W(t) \mid X(t) = x]$

$+ 2 E[\ln X(t) \mid X(t) = x]$

$= 2(\mu - \frac{1}{2}\sigma^2)(T-t) + 2\sigma E(W(T) - W(t)) + 2\ln x$

ie:

$F(t, x) = 2(\mu - \frac{1}{2}\sigma^2)(T-t) + 2\ln x \quad \forall t \leq T$

$F(T, x) = 2(\mu - \frac{1}{2}\sigma^2)(T-T) + 2\ln x = \ln x^2$

4. Solve the following boundary value problem:

$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,x) = 0$$
$$F(T,x) = x^2 - x$$

$\mu = 0$
$\sigma = 1$

where σ is a known constant.

following partial differential equation:

$$\frac{\partial F(\tau,x)}{\partial \tau} + \mu(\quad x)\frac{\partial F(\tau,x)}{\partial x} + \frac{1}{2}\sigma^2(\quad x)\frac{\partial^2 F(\tau,x)}{\partial x^2} = 0 \qquad (*)$$

$$dX(\tau) = 0\,d\tau + 1\,dW(\tau) \Rightarrow dX(\tau) = dW(\tau)$$

$$F(t,x) = E[\,W^2(T) - W(T) \mid W(t) = x\,]$$

$$= E[\,(W(T) - W(t) + W(t))^2 - (W(T) - W(t) + W(t)) \mid W(t) = x\,]$$

$$= E[\,(W(T) - W(t))^2 + W^2(t) + 2(W(T) - W(t))W(t) \mid W(t) = x\,]$$

$$- E[\,W(T) - W(t) \mid W(t) = x\,] - E[\,W(t) \mid W(t) = x\,]$$

$$= E[(W(T) - W(t))^2] + x^2 + 2x\,E[W(T) - W(t)]$$

$$- E[W(T) - W(t)] - x$$

$$= \mathrm{var}(W(T) - W(t)) + x^2 + 0 - 0 - x$$

$$= (T - t) + x^2 - x \quad //$$

$$\frac{1}{2}\sigma^2 = 2 \Rightarrow \sigma^2 = 4 \Rightarrow \sigma = 2$$

- Find F such that: $$\frac{\partial F}{\partial t} + 2\frac{\partial^2 F}{\partial x^2} = 0$$
$$F(T,x) = x$$

$\mu = 0$; $\sigma = 2$ $\quad \Rightarrow X(t) = 2W(t)$

$$dX(t) = 0\,dt + 2\,dW(t) \Rightarrow dX(t) = 2\,dW(t)$$

$$F(t,x) = E[\,2W(T) \mid W(t) = \tfrac{x}{2}\,] = 2E[W(T) - W(t)$$

$$+ W(t) \mid W(t) = \tfrac{x}{2}\,] = 2\underbrace{E[W(T) - W(t)]}_{\text{independence of increments}} + 2E[W(t) \mid W(t) = \tfrac{x}{2}]$$

$= 2\times 0 + 2\frac{x}{2} = 2\frac{x}{2} = x \quad , \quad \forall\, \tau \in [0, T]$

$\mu = x \qquad \frac{1}{2}\sigma^2 = x^2$

$$\frac{\partial F}{\partial \tau} + x\frac{\partial F}{\partial x} + x^2 \frac{\partial^2 F}{\partial x^2} = 0$$

$$F(T, x) = \ln x + T$$

$$\boxed{dX(\tau) = \mu X(\tau)\,d\tau + \underbrace{\sqrt{2}\,X(\tau)}^{\sigma X(\tau)}\,dW(\tau)}$$

$$\Rightarrow X(T) = X(\tau)\,e^{\left(1-\frac{1}{2}\times 2\right)(T-\tau) + \sqrt{2}(W(T)-W(\tau))} = X(\tau)\exp\left(\sqrt{2}\,(W(T)-W(\tau))\right)$$

$$F(\tau, x) = E[\ln X(T) + T \mid X(\tau) = x]$$

$$= E[\ln X(\tau) + \sqrt{2}\,(W(T)-W(\tau)) + T \mid X(\tau) = x]$$

$$= \ln x + T + \sqrt{2}\,E[\cancel{W(T) - W(\tau)} \mid W(\tau)] \quad (=0)$$

$$= \ln x + T$$

$$dX(\tau) = \mu X(\tau)\,d\tau + \sigma X(\tau)\,dW(\tau)$$

$$\Rightarrow X(T) = X(\tau)\,e^{\left(\mu-\frac{1}{2}\sigma^2\right)(T-\tau) + \sigma(W(T)-W(\tau))}$$

Compute the solution of such problem.

6. Solve the following boundary value problem:

$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2}(t,x) = 0, \quad t \in [0,T], x > 0$$
$$F(T,x) = (\ln(x))^2$$